

Some aspects of triangular norms

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Introduction

The main objective of this work is to improve our knowledge about the notions of triangular norms on a bounded lattice. Some specific objectives are: - Study some basic algebraic aspects. - Study some basic analytic aspects. - Study some basic logic aspects. Finally, as application of triangular norms, we have tried to understand the relations obtained from a triangular norm. For this we have consider some paper in this subject [7, 8].

Although the notion triangular norm appeared already in Karl Menger's paper "Statistical metrics" in 1942 [14], the interest in probabilistic metric spaces in general and in these operations in particular started to increase only after Berthold Schweizer and Abe Sklar published the final set of axioms for t-norms in 1958 in "*Espaces métriques alméatoires*" [17].

In the sequel, triangular norms turned out to be basic tools not only in probabilistic metric spaces (as they are called today), where they allowed the classical triangle inequality to be properly generalized, but also in several other parts of mathematics, initiating a deeper investigation of t-norms from different points of view.

Starting from the first Linz Seminar on Fuzzy Set Theory in 1979, when Ulrich Höhle [5] suggested to base the intersection and union of fuzzy subsets on triangular norms and conorms.

Triangular norms and conorms are operations which generalize the logical conjunction and logical disjunction to fuzzy logic. They are a natural interpretation of the conjunction and disjunction in the semantics of mathematical fuzzy logics. Triangular norms are an indispensable tool for the interpretation of the conjunction in fuzzy logics and, subsequently, for the intersection of fuzzy sets; They are, however, interesting mathematical objects for themselves; Triangular norms, as we use them today, They also play an important role in decision making.

The memory is divided into three chapters. In the first Chapter, we give some basic notions and definitions of ordered sets, lattices and fuzzy ordered sets needed in the sequel. Also, we focus on some results concerning the binary operations on the unite real interval $[0, 1]$ and we

give some properties and characterizations of binary operations on the unit interval $[0, 1]$.

The second chapter is devoted to study triangular norms and triangular conorms, where some basic definitions and properties were given. Also, elementary algebraic properties, additive and multiplicative generator were given.

In Chapter 3, as application of the concept of triangular norm, we study triangular orders, i.e., t-partial orders obtained from t-norms. Also, we study some particular sets of this structure.

Chapter 1

Preliminaries

In this chapter, we recall some useful notions on order structures, lattices and we give some properties and characterizations of binary operations on the unit interval $[0, 1]$.

1.1 Basic concepts

1.1.1 Binary Relation

In this section, we recall some useful notions on order structures, lattices. For further details, we refer to [1, 3].

Definition 1.1 (Cartesian Products). [15]

The Cartesian product of two sets S and T is generally defined as the set

$$S \times T = \{(s, t) \mid s \in S, t \in T\}$$

Definition 1.2 (Binary Relations). [15]

Let A be a nonempty set. A binary relation on A is a subset R of the Cartesian product $A \times A$. We write $(a, b) \in R$ as $a \sim b$. A binary relation is

1) **reflexive** if for all $a \in A$,

$$a \sim a$$

2) **irreflexive** if for all $a \in A$,

$$a \approx a$$

3) **symmetric** if for all $a, b \in A$,

$$a \sim b \Rightarrow b \sim a$$

4) **asymmetric** if for all $a, b \in A$,

$$a \sim b \Rightarrow b \approx a$$

5) **antisymmetric** if for all $a, b \in A$,

$$a \sim b, b \sim a \Rightarrow b = a$$

6) **transitive** if for all $a, b \in A$,

$$a \sim b, b \sim c \Rightarrow a \sim c$$

1.1.2 Partial order

Definition 1.3 (Partial order). [3]

Let P be a set. An order (or partial order) on P is a binary relation $\leq$ on P such that $\leq$ is

(i) reflexive;

(ii) antisymmetric;

(iii) transitive.

A set P equipped with an order relation $\leq$ is said to be an ordered set (or partially ordered set).

Some authors use the shorthand poset.

Definition 1.4 (Strict order). [15]

A binary relation $<$ on a nonempty set P is called a strict partial order or (strict order) if it is asymmetric and transitive.

Given a partially ordered set $(P, \leq)$, we can define a strict order $<$ by

$$x < y \text{ if } x \leq y \text{ and } x \neq y$$

Example 1.1.

1. The lexicographic order on the Cartesian product $P \times Q$ is defined by

$$(p_1, q_1) \leq (p_2, q_2) \text{ if } p_1 < p_2 \text{ or } (p_1 = p_2 \text{ and } q_1 \leq q_2)$$

This is a partial order on $P \times Q$.

2. Let $N = \{0, 1, \dots\}$ be the set of natural numbers. Then $(N, \leq)$ is also a poset under the ordinary order.
3. Any family of sets is ordered by set inclusion $\subseteq$.
4. The natural numbers can also be ordered as follows: For $p, q \in N$ we say $p \leq q$ iff p divides q .

Definition 1.5 (Comparability). [16]

The elements a and b of a poset $(P, \leq)$ are called *comparable* if either $a \leq b$ or $b \leq a$. When a and b are elements of P such that neither $a \leq b$ nor $b \leq a$, a and b are called *incomparable*.

Definition 1.6.

A partial order relation $\leq$ on A is called a **linear order** if for each $a, b \in A$ either $a \leq b$ or $b \leq a$. In this case, $(A, \leq)$ is called a **linearly ordered set**, or a **chain**.

1.1.3 Lattice**Definition 1.7.** [12]

Let $(A, \leq)$ be a poset and $B \subseteq A$.

- (i) $a \in A$ is called an **upper bound** of B if $b \leq a$ for all $b \in B$.
- (ii) $a \in A$ is called a **lower bound** of B if $a \leq b$ for all $b \in B$.
- (iii) The greatest amongst the lower bounds of B , whenever it exists, is called the **infimum** of B , and is denoted by $\inf B$.
- (iv) The least upper bound of B , whenever it exists, is called the **supremum** of B , and is denoted by $\sup B$.

Notation 1.1.

Looking ahead, we shall adopt the following neater notation: we write $x \vee y$ (read as x join y) in place of $\sup\{x, y\}$ when it exists and $x \wedge y$ (read as x meet y) in place of $\inf\{x, y\}$ when it exists.

Definition 1.8. [12]

A poset $(L, \leq)$ is called **lattice ordered** if for every pair x, y of elements of L their supremum and infimum exist.

Definition 1.9.

A subset S of a lattice L is called a **sublattice** of L if S is a lattice with respect to the restrictions of $\wedge$ and $\vee$ from L to S ; (i.e., $s_1, s_2 \in S \Rightarrow s_1 \wedge s_2 \in S$ and $s_1 \vee s_2 \in S$).

Example 1.2.

(i) Every singleton of a lattice L is a sublattice of L .

(ii) For any two elements x, y in a lattice L , the interval

$$[x, y] := \{a \in L \mid x \leq a \leq y\} \quad (1.1)$$

is a sublattice of L .

(iii) Let L be the lattice of all subsets of a vector space V and let S be the set of all subspaces of V . Then S is a lattice with respect to inclusion but not a sublattice of L .

1.2 Binary operations on the unit interval $[0, 1]$

In this section, we recall definitions and properties of binary operations on the unit interval $[0, 1]$ [18].

1.2.1 Definitions and properties

Definition 1.10.

A **binary operation** F on the unit interval is a function $F : [0, 1] \times [0, 1] \rightarrow [0, 1]$. The following properties of binary operations are of interest in this work:

(i) *commutativity*: $F(x, y) = F(y, x)$, for any $x, y \in [0, 1]$;

(ii) *associativity*: $F(x, F(y, z)) = F(F(x, y), z)$, for any $x, y, z \in [0, 1]$;

An element $e \in [0, 1]$ is called a

(iii) *left-neutral (resp. right-neutral) element* of F , if $F(e, x) = x$ (resp. $F(x, e) = x$), for any $x \in [0, 1]$;

(iv) *neutral element* of F , if it is left- and right-neutral element of F , i.e., $F(e, x) = F(x, e) = x$, for any $x \in [0, 1]$.

Example 1.3.

(1) $F(x, y) = |\sin(x + y)|$.

(2) $F(x, y) = \frac{\exp x + y}{\exp 2}$.

(3) $F(x, y) = xy$.

(4) $F(x, y) = \frac{x + y}{2}$.

(5) $F(x, y) = \frac{\ln x + y + 1}{\ln 3}$.

For further details, we refer to [4, 11, 12, 13].

Definition 1.11. A binary operation F on a sublattice of the unit interval $[0, 1]$ ($L, \leq, \min, \max$) is called:

(i) *idempotent*, if $F(x, x) = x$, for any $x \in L$;

(ii) *conjunctive*, if $F(x, y) \leq \min(x, y)$, for any $x, y \in L$;

(iii) *disjunctive*, if $\max(x, y) \leq F(x, y)$, for any $x, y \in L$;

- (iv) *left-increasing*, if $x \leq y$, implies $F(x, z) \leq F(y, z)$, for any $x, y, z \in L$;
- (v) *right-increasing*, if $x \leq y$, implies $F(z, x) \leq F(z, y)$, for any $x, y, z \in L$;
- (vi) *increasing*, if $x_1 \leq x_2$ and $y_1 \leq y_2$ imply $F(x_1, y_1) \leq F(x_2, y_2)$, for any $x_1, y_1, x_2, y_2 \in L$.

We note that a binary operation F on the unit interval $[0, 1]$ is increasing if and only if it is left- and right-increasing. Also, if F is commutative, then it holds that F is left-increasing if and only if F is right-increasing.

Example 1.4.

Let $(L, \leq, \min, \max)$ be a sublattice of the unit interval $[0, 1]$. The binary operations $\min$ and $\max$ are idempotent and increasing. Moreover, $\min$ is conjunctive and $\max$ is disjunctive.

Definition 1.12. Let $(L, \leq, \min, \max)$ be a sublattice of the unit interval $[0, 1]$ and F be a binary operation on L . The binary operation F^t defined as $F^t(x, y) = F(y, x)$, for any $x, y \in L$, is called the transpose of F .

The following proposition expresses the relationships between the properties of a binary operation and those of its transpose. The proof is straightforward.

Proposition 1.1. [18]

Let $(L, \leq, \min, \max)$ be a sublattice of the unit interval $[0, 1]$ and F be a binary operation on L . The following equivalences hold:

- (i) F is commutative if and only if $F^t = F$;
- (ii) F is conjunctive if and only if F^t is conjunctive;
- (iii) F is disjunctive if and only if F^t is disjunctive;
- (iv) F is left-increasing if and only if F^t is right-increasing;
- (v) F is increasing if and only if F^t is increasing;
- (vi) An element $e \in L$ is a left-neutral element of F if and only if e is a right-neutral element of F^t ;
- (vii) An element $e \in L$ is a neutral element of F if and only if e is a neutral element of F^t .

1.2.2 Characterizations of some properties of binary operations on a sublattice of the unit interval $[0, 1]$

In this subsection, we study and characterize some properties of a binary operation on a sublattice of the unit interval $[0, 1]$.

Proposition 1.2.

Let $(L, \leq, \min, \max)$ be a sublattice of the unit interval $[0, 1]$ and F be a binary operation on L . Then it holds that

- (i) *If F is increasing, idempotent and conjunctive, then $\min(F(x, y), \alpha) = F(\min(x, \alpha), \min(y, \alpha)) = F(\min(x, \alpha), y) = F(x, \min(y, \alpha))$ for any $x, y, \alpha \in L$;*
- (ii) *If F is increasing, idempotent and disjunctive, then $\max(F(x, y), \alpha) = F(\max(x, \alpha), \max(y, \alpha)) = F(\max(x, \alpha), y) = F(x, \max(y, \alpha))$ for any $x, y, \alpha \in L$.*

Proof.

Let $x, y, \alpha \in L$.

- (i) We only prove that $\min(F(x, y), \alpha) = F(\min(x, \alpha), \min(y, \alpha))$, as the other equalities can be proved similarly. Since F is increasing, it follows that $F(\min(x, \alpha), \min(y, \alpha)) \leq F(x, y)$. The fact that F is conjunctive implies that $F(\min(x, \alpha), \min(y, \alpha)) \leq \alpha$. Hence, $F(\min(x, \alpha), \min(y, \alpha))$ is a lower bound of $\{F(x, y), \alpha\}$. On the other hand, let $m \in L$ another lower bound of $\{F(x, y), \alpha\}$, i.e., $m \leq F(x, y)$ and $m \leq \alpha$. Since F is conjunctive, it follows that $m \leq x$ and $m \leq y$. Hence, $m \leq \min(x, \alpha)$ and $m \leq \min(y, \alpha)$. Since F is idempotent and increasing, it holds that $m = F(m, m) \leq F(\min(x, \alpha), \min(y, \alpha))$. Thus, $F(\min(x, \alpha), \min(y, \alpha))$ is the greatest lower bound of $\{F(x, y), \alpha\}$, i.e., $\min(F(x, y), \alpha) = F(\min(x, \alpha), \min(y, \alpha))$.

- (ii) The proof is analogous to that of (i).

□

The following lemma generalizes a remarkable result of Cooman and Kerre (Proposition 3.7 in [2]) to a sublattice of the unit interval $[0, 1]$.

Lemma 1.1.

Let $(L, \leq, \min, \max)$ be a sublattice of the unit interval $[0, 1]$. If an idempotent binary operation F on L is increasing and conjunctive (resp. disjunctive), then F can not be other than the meet operation $\min$ (resp. the join operation $\max$) of L .

Proof.

Suppose that F is idempotent, increasing and conjunctive binary operation on L . From Proposition 1.2 (i) it follows that $\min(F(x, y), \min(x, y)) = F(\min(x, \min(x, y), \min(y, \min(x, y))) = F(\min(x, y), \min(x, y)) = \min(x, y)$, for any $x, y \in L$. Hence, $\min(x, y) \leq F(x, y)$, for any $x, y \in L$. Now, the fact that F is conjunctive guarantees that $F(x, y) = \min(x, y)$, for any $x, y \in L$. Thus, $F = \min$.

In a similar way, we prove that F is the join operation $\max$ of L .

□

The following proposition shows an equivalent condition to both properties of increasing and conjunctive binary operation on a sublattice of the unit interval $[0, 1]$.

Proposition 1.3.

Let $(L, \leq, \min, \max)$ be a sublattice of the unit interval $[0, 1]$ and F be a binary operation on L . If F is idempotent, then the following conditions are equivalent:

- (i) F is increasing and conjunctive;
- (ii) $\min(F(x, y), \alpha) = F(\min(x, \alpha), \min(y, \alpha)) = F(\min(x, \alpha), y) = F(x, \min(y, \alpha))$, for any $x, y, \alpha \in L$.

Proof.

(i) $\Rightarrow$ (ii): The proof is directly from Proposition 1.2.

(ii) $\Rightarrow$ (i): First, we prove that F is conjunctive. Let $x, y \in L$, from (ii) it follows that $\min(F(x, y), x) = F(\min(x, x), y) = F(x, y)$ and $\min(F(x, y), y) = F(x, \min(y, y)) = F(x, y)$. Hence, $F(x, y) \leq x$ and $F(x, y) \leq y$. Thus, $F(x, y) \leq \min(x, y)$. Consequently, F is conjunctive. Second, we prove that F is increasing. Let $x, y, t, z \in L$ such that $x \leq y$. Since F is idempotent, it follows that $\min(F(x, z), F(y, z)) = F(\min(F(x, z), y), \min(F(x, z), z)) = F(F(\min(x, y), z), F(x, \min(z, z))) = F(F(x, z), F(x, z)) = F(x, z)$. Hence, $F(x, z) \leq F(y, z)$. Thus, F is left-increasing. Moreover, $\min(F(t, x), F(t, y)) = F(\min(F(t, x), t), \min(F(t, x), y)) =$

$F(F(\min(t, t), x), F(t, \min(x, y))) = F(F(t, x), F(t, x)) = F(t, x)$. This implies that $F(t, x) \leq F(t, y)$. Hence, F is right-increasing. Now, since F is left- and right-increasing, it holds that F is increasing. $\square$

In the same line, we obtain the following proposition.

Proposition 1.4.

Let $(L, \leq, \min, \max)$ be a sublattice of the unit interval $[0, 1]$ and F be a binary operation on L . If F is idempotent, then the following conditions are equivalent:

- (i) *F is increasing and disjunctive;*
- (ii) *$\max(F(x, y), \alpha) = F(\max(x, \alpha), \max(y, \alpha)) = F(\max(x, \alpha), y) = F(x, \max(y, \alpha))$, for any $x, y, \alpha \in L$.*

The following theorem establishes a simple criterion to check, if a binary operation on a given sublattice of the unit interval $[0, 1]$ coincides with its meet($\min$) (resp. its join($\max$)) operation.

Theorem 1.1. *Let $(L, \leq, \min, \max)$ be a sublattice of a unit interval $[0, 1]$ and F be an idempotent binary operation on L . Then it holds that*

- (i) *F is the meet operation $\min$ if and only if $\min(F(x, y), \alpha) = F(\min(x, \alpha), \min(y, \alpha)) = F(\min(x, \alpha), y) = F(x, \min(y, \alpha))$, for any $x, y, \alpha \in L$;*
- (ii) *F is the join operation $\max$ if and only if $\max(F(x, y), \alpha) = F(\max(x, \alpha), \max(y, \alpha)) = F(\max(x, \alpha), y) = F(x, \max(y, \alpha))$, for any $x, y, \alpha \in L$.*

Proof.

The proof can be easily obtained from Lemma 1.1, Propositions 1.3 and 1.4. $\square$

Next, we need the following remark.

Remark 1.1.

In view of Lemma 1.1, if a binary operation F on a sublattice $(L, \leq, \min, \max)$ of the unit interval $[0, 1]$ is idempotent, conjunctive (resp. disjunctive) and increasing, then $F = \min$ (resp. $F = \max$). But, if F is only left- or right-increasing, then F is not necessarily the meet (resp.

the join) operation of L . Indeed, let $(L = \{0, 1/2, 1\}, \leq, \min, \max)$ be the sublattice of the unit interval given by the Hasse diagram in Fig. 1.1, and F, G be two binary operations on L defined by the following tables:

$F(x, y)$	0	1/2	1
0	0	0	0
1/2	0	1/2	0
1	0	1/2	1

$G(x, y)$	0	1/2	1
0	0	1/2	1
1/2	1	1/2	1
1	1	1	1

One can easily verify that F is left-increasing, conjunctive and idempotent, but it is not the $\min$ operation of L . Analogously, F^t is right-increasing, conjunctive and idempotent, but also it is not the $\min$ operation of L . Moreover, the binary operation G (resp. G^t) is left-increasing (resp. right-increasing), disjunctive and idempotent, but it is not the $\max$ operation of L .



Figure 1.1: The Hasse diagram of the lattice $(L = \{0, \frac{1}{2}, 1\} \leq)$

Chapter 2

Triangular norms and triangular conorms

In fuzzy sets theory, The conjunction is usually interpreted by a triangular norm and the disjunction by a triangular conorm.

2.1 Triangular norms

In this section, we recall some basic definitions and notions concerning triangular norms.

2.1.1 Basic definitions and properties

Definition 2.1. [10] *A triangular norm (t-norm for short) is a binary operation T on the unit interval $[0, 1]$, i.e., it is a function $T : [0, 1]^2 \rightarrow [0, 1]$ such that for all $x, y, z \in [0, 1]$: the following four axioms are satisfied:*

(T1) *Commutativity:* $T(x, y) = T(y, x)$;

(T2) *Associativity:* $T(x, T(y, z)) = T(T(x, y), z)$;

(T3) *Monotonicity:* $T(x, y) \leq T(x, z)$ whenever $y \leq z$;

(T4) *Boundary condition:* $T(x, 1) = x$.

Example 2.1.

The following are the four basic t-norms T_M , T_P , T_L , and T_D given respectively by:

$T_M(x, y) = \min(x, y)$	(Minimum) Figure 2.1
$T_P(x, y) = x \cdot y$	(Product) Figure 2.2
$T_L(x, y) = \max(x + y - 1, 0)$	(Łukasiewicz t-norm) Figure 2.3
$T_D(x, y) = \begin{cases} 0 & \text{if } (x, y) \in [0, 1]^2 \\ \min(x, y) & \text{otherwise} \end{cases}$	(Drastic product) Figure 2.4

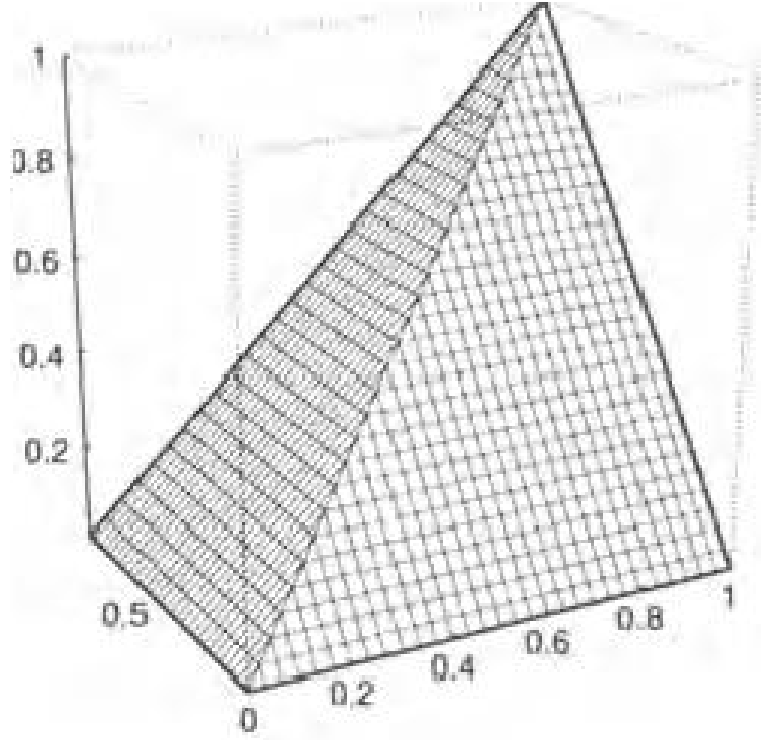


Figure 2.1: Minimum t-norm

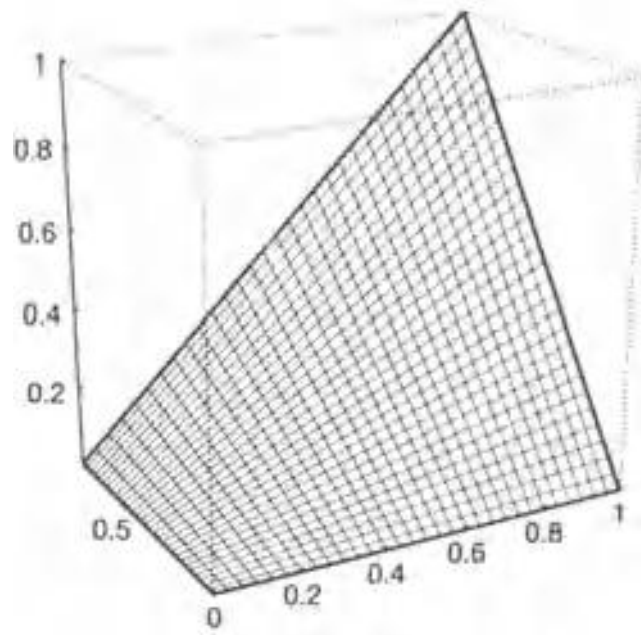


Figure 2.2: Product t-norm

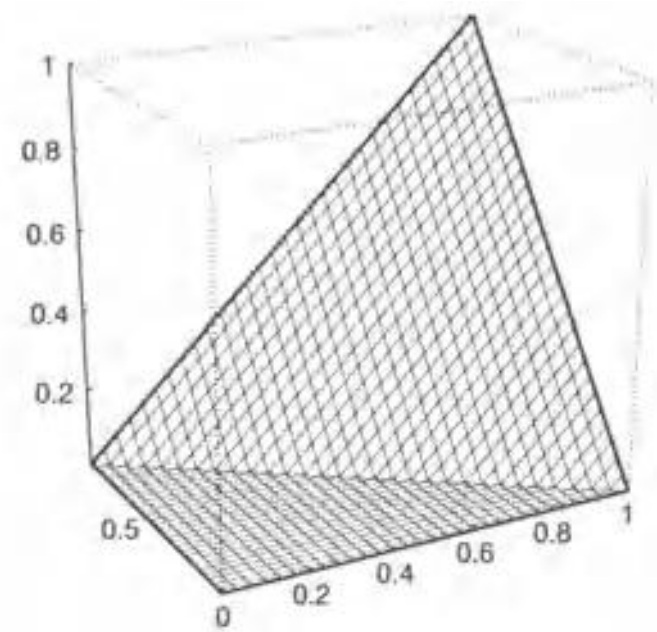


Figure 2.3: Łukasiewicz t-norm

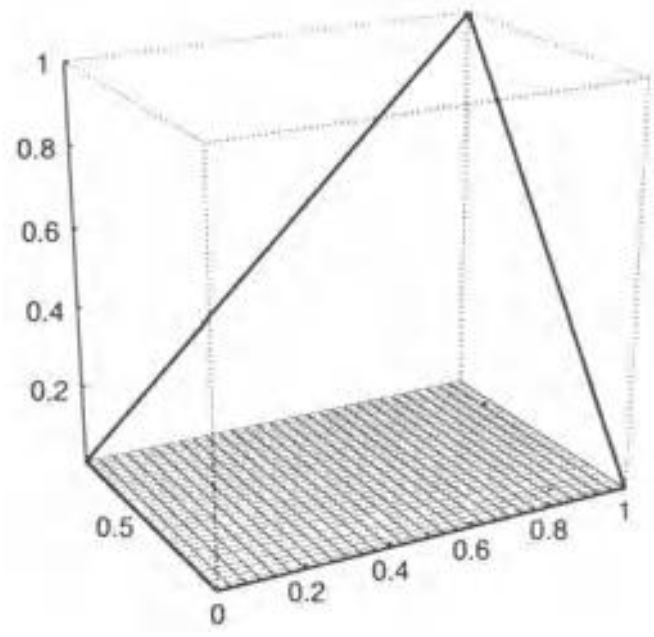


Figure 2.4: Drastic Product t-norm

Example 2.2.

$T(x, y) = \frac{xy}{(2 - x - y + xy)}$	<i>Einstein</i>
$T(x, y) = \frac{xy}{(x + y - xy)}$	<i>Hamacher</i>
$T(x, y) = \frac{xy}{\max(x, y, \alpha)}$	<i>Dubois and Parade (1986) $\alpha \in [0, 1]$</i>

Proposition 2.1.

Any t-norm T satisfies $T(0, x) = T(x, 0) = 0$, for all $x \in [0, 1]$.

Proof.

We know that $T(x, 0) \in [0, 1]$, so $T(x, 0) \geq 0$, and we use the axiom (T3)(monotonicity), we obtain $T(x, 0) \leq T(1, 0) = 0$. $\square$

Proposition 2.2.

Let A be a set with $]0, 1[\subseteq A \subseteq [0, 1]$, and assume that $F : A^2 \rightarrow A$ is a binary operation on A such that for all $x, y, z \in A$ the properties (T1) - (T3) and

$$F(x, y) \leq \min(x, y) \quad (*)$$

are satisfied. Then the function $T : [0, 1]^2 \rightarrow [0, 1]$ defined by

$$T(x, y) = \begin{cases} F(x, y) & \text{if } (x, y) \in (A \setminus \{1\})^2, \\ \min(x, y) & \text{otherwise.} \end{cases}$$

is a t -norm.

Proof.

The commutativity (T1) and the boundary condition (T4) are satisfied by definition. Concerning the associativity (T2), observe that for $x, y, z \in A \setminus \{0, 1\}$ we have $T(T(x, y), z) = T(x, T(y, z))$ as a consequence of the associativity of F . If $0 \in \{x, y, z\}$ then we get $T(x, T(y, z)) = 0 = T(T(x, y), z)$, and if $1 \in \{x, y, z\}$ then $T(T(x, y), z) = T(x, T(y, z))$ follows from (T4). Concerning the monotonicity (T3), suppose $y \leq z$. In the cases $x, y, z \in A \setminus \{1\}$ or $x \in \{0, 1\}$ or $y = 0$, the inequality $T(x, y) \leq T(x, z)$ is inherited from the monotonicity of F and $\min$. The only non-trivial case is when $x, y \in A \setminus \{1\}$ and $z = 1$, in which case $T(x, y) \leq T(x, z)$ follows from (*). $\square$

Definition 2.2. [9] A function $f : [0, 1]^2 \rightarrow [0, 1]$ which satisfies, for all $x, y, z \in [0, 1]$, the properties (T1)- (T3) and $f(x, y) \leq \min(x, y)$ is called a t -subnorm.

Example 2.3.

$$1- f(x, y) = 0.$$

$$2- f(x, y) = \frac{x \cdot y}{3}.$$

$$3- f(x, y) = x \cdot y.$$

Remark 2.1.

Clearly, each t -norm is a t -subnorm, but not vice versa: for example, the function $f : [0, 1]^2 \rightarrow [0, 1]$ given by $f(x, y) = 0$, is a t -subnorm but not a t -norm because (T4) not satisfies ($f(x, 1) = 0 \neq x$).

Corollary 2.1. [10]

If f is a t -subnorm then the function $T : [0, 1]^2 \rightarrow [0, 1]$ defined by

$$T(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \in [0, 1]^2, \\ \min(x, y) & \text{otherwise,} \end{cases}$$

is a triangular norm.

Comparison of t-norms

Definition 2.3. [10]

- (i) If, for two t-norms T_1 and T_2 , the inequality $T_1(x, y) \leq T_2(x, y)$ holds for all $(x, y) \in [0, 1]^2$, then we say that T_1 is weaker than T_2 or, equivalently, that T_2 is stronger than T_1 , and we write in this case $T_1 \leq T_2$.
- (ii) We shall write $T_1 < T_2$ whenever $T_1 \leq T_2$ and $T_1 \neq T_2$, i.e., if $T_1 \leq T_2$ and for some $(x_0, y_0) \in [0, 1]^2$ we have $T_1(x_0, y_0) < T_2(x_0, y_0)$

Lemma 2.1.

- (i) The minimum T_M is the strongest t-norm ($T_M \geq T$).
- (ii) The drastic product T_D is the weakest t-norm ($T_D \leq T$).

Proof.

- (i) For each t-norm T and for each $(x, y) \in [0, 1]^2$ we have both $T(x, y) \leq T(x, 1) = x$ and $T(x, y) \leq T(1, y) = y$, so $T(x, y) \leq \min(x, y) = T_M(x, y)$.
- (ii) All t-norms coincide on the boundary of $[0, 1]^2$ and for all $(x, y) \in]0, 1[^2$ we trivially have $T(x, y) \geq 0 = T_D(x, y)$.

□

Example 2.4.

- $T_0(x, y) = \begin{cases} 0 & \text{if } (x, y) \in [0, 1[^2, \\ \min(x, y) & \text{otherwise,} \end{cases} \quad (\text{Drastic product of weber}).$
- $T_1(x, y) = \max(x + y - 1, 0) \quad (\text{\Lukasiewicz}).$
- $T_{1.5}(x, y) = \frac{xy}{2 - x - y + xy} \quad (\text{Einstein}).$
- $T_2(x, y) = xy \quad (\text{Algebraic or probabilistic}).$
- $T_{2.5}(x, y) = \frac{xy}{x + y - xy} \quad (\text{Hamacher}).$

$$- T_3(x, y) = \min(x, y) \quad (\text{Zadeh}).$$

we have: $T_0 \leq T_1 \leq T_{1.5} \leq T_2 \leq T_{2.5} \leq T_3$.

Definition 2.4. [10] (Domination of t -norm)

Let T_1 and T_2 be two t -norms. Then we say that T_1 dominates T_2 (in symbols $T_1 \gg T_2$) if for all $x, y, u, v \in [0, 1]$

$$T_1(T_2(x, y), T_2(u, v)) \geq T_2(T_1(x, u), T_1(y, v)). \quad ((\text{Equ 1}))$$

Lemma 2.2.

(i) For each t -norm T , we have $T_M \gg T$ and $T \gg T_D$.

(ii) If for two t -norms T_1 and T_2 we have T_1 dominates T_2 ($T_1 \gg T_2$) then, T_1 is stronger than T_2 ($T_1 \geq T_2$).

(iii) The relation $\gg$ on the set of all t -norms is reflexive and antisymmetric.

Proof.

(i) Trivially, (by separation of cases)

(ii) If for two t -norms T_1 and T_2 we have $T_1 \gg T_2$ then, putting $y = u = 1$ in (Equ 1), we immediately see that also $T_1 \geq T_2$ holds.

(iii) from the commutativity (T_1) and the associativity (T_2) we obtain for each t -norm T and all $x, y, u, v \in [0, 1]$ the equality

$$T(T(x, y), T(u, v)) = T(T(x, u), T(y, v))$$

car

$$\begin{aligned} T(T(x, y), T(u, v)) &= T(x, T(y, T(u, v))) \\ &= T(x, T(T(y, u), v)) \\ &= T(x, T(T(u, y), v)) \\ &= T(x, T(u, T(y, v))) \\ &= T(T(x, u), T(y, v)) \end{aligned}$$

i.e., $T \gg T$, and the assumptions $T_1 \gg T_2$ and $T_2 \gg T_1$ imply, as a consequence of (ii), $T_1 = T_2$.

□

Remark 2.2.

The converse is false: $T_1 \geq T_2$ does not imply $T_1 \gg T_2$.

consider the t -norm T_P and the t norm T given by:

$$T(x, y) = \begin{cases} \frac{xy}{2} & \text{if } (x, y) \in [0, 1]^2, \\ \min(x, y) & \text{otherwise,} \end{cases}$$

we have $T_P \geq T$ but $T_P \gg T$ is false.

let $(x, y) \in [0, 1]^2$ if $(x, y) \in [0, 1]^2$ hence $T_P = xy > \frac{xy}{2} = T(x, y)$

if $\max(x, y) = 1$ $T_P(x, y) = \min(x, y) = T(x, y)$.

So $\forall (x, y) \in [0, 1]^2$ we have $T_P(x, y) \geq T(x, y)$ i.e., $T_P \geq T$

but $T_P(T(x, y), T(u, v)) \not\geq T(T_P(x, u), T_P(y, v))$,

because if $(x, y) \in [0, 1]^2$ and $(u, v) \in [0, 1]^2$ we get $T_P(T(x, y), T(u, v)) = \frac{xyuv}{4}$ and $T(T_P(x, u), T_P(y, v)) = \frac{xyuv}{2}$.

Proposition 2.3.

(i) The only t -norm T satisfying $T(x, x) = x$ for all $x \in [0, 1]$ is the minimum T_M .

(ii) The only t -norm T satisfying $T(x, x) = 0$ for all $x \in [0, 1[$ is the drastic product T_D .

Proof.

(i) If for a t -norm T we have $T(x, x) = x$ for each $x \in [0, 1]$, then for all $(x, y) \in [0, 1]^2$ with $y \leq x$ the monotonicity (T3) implies $y = T(y, y) \leq T(x, y) \leq T_M(x, y) = y$,

which, together with (T1), means $T = T_M$.

(ii) Assume $T(x, x) = 0$ for each $x \in [0, 1[$. Then for all $(x, y) \in [0, 1]^2$ with $y \leq x$ we have $0 \leq T(x, y) \leq T(x, x) = 0$, hence, together with (T1) and (T4), yielding $T = T_D$.

□

Continuity

Definition 2.5. [9] A T -norm $T : [0, 1]^2 \rightarrow [0, 1]$ is continuous if for all the convergent sequences $(x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}} \in [0, 1]^{\mathbb{N}}$ we have:

$$T\left(\lim_{n \rightarrow \infty} x_n, \lim_{n \rightarrow \infty} y_n\right) = \lim_{n \rightarrow \infty} T(x_n, y_n).$$

Definition 2.6. A t -norm $T : [0, 1]^2 \rightarrow [0, 1]$ is said to be left-continuous (right-continuous) if for each $y \in [0, 1]$ and for all non-decreasing (non-increasing) sequences $(x_n)_{n \in \mathbb{N}}$ we have :

$$\lim_{n \rightarrow \infty} T(x_n, y) = T\left(\lim_{n \rightarrow \infty} x_n, y\right)$$

Remark 2.3.

A t -norm is continuous if and only if it is both left- and right-continuous

Example 2.5.

1/ The t -norm given by:

$$T(x, y) = \begin{cases} 0 & \text{if } x + y \leq 1, \\ \min(x, y) & \text{otherwise,} \end{cases}$$

is left-continuous but not right-continuous.

2/ The drastic product T_D , is right-continuous but not left-continuous.

3/ A t -norm which is neither left- nor right-continuous:

$$T(x, y) = \begin{cases} \frac{xy}{2} & \text{if } (x, y) \in [0, 1]^2 \\ \min(x, y) & \text{otherwise} \end{cases}$$

2.1.2 Elementary algebraic properties

Definition 2.7. [9]

(i) An element $a \in [0, 1]$ is called an idempotent element of T if $T(a, a) = a$.

The numbers 0 and 1 (which are idempotent elements for each t -norm T) are called trivial idempotent elements of T , each idempotent element in $]0, 1[$ will be called a non-trivial idempotent element of T .

- (ii) An element $a \in]0, 1[$ is called a nilpotent element of T if there exists some $n \in \mathbb{N}$ such that $a_T^{(n)} = 0$.
- (iii) An element $a \in]0, 1[$ is called a zero divisor of T if there exists some $b \in]0, 1[$ such that $T(a, b) = 0$.

Example 2.6.

- (i) Each $a \in [0, 1]$ is an idempotent element of the minimum T_M (actually, T_M is the only t -norm whose set of idempotent elements equals $[0, 1]$).
- (ii) Each $a \in]0, 1[$ is both a nilpotent element and a zero divisor of the Lukasiewicz t -norm T_L as well as of the drastic product T_D .
- (iii) The minimum T_M has neither nilpotent elements nor zero divisors, and T_L and T_D both possess only trivial idempotent elements. The product T_P has neither non-trivial idempotent nor nilpotent elements nor zero divisors.
- (iv) For the nilpotent minimum T^{nM} defined par

$$T(x, y) = \begin{cases} 0 & \text{if } x + y \leq 1, \\ \min(x, y) & \text{otherwise,} \end{cases}$$

a number a is an idempotent element if and only if $a \in \{0\} \cup]0.5, 1]$, a is a nilpotent element if and only if $a \in]0, 0.5]$, and a is a zero divisor if and only if $a \in]0, 1[$.

Remark 2.4. [9]

For arbitrary t -norms some general observations concerning idempotent and nilpotent elements and zero divisors can be formulated.

- (i) If $a \in [0, 1]$ is an idempotent element of a t -norm T then, by induction, we also have $a_T^{(n)} = a$ for all $n \in \mathbb{N}$. In particular, this means that no element of $]0, 1[$ can be both idempotent and nilpotent.
- (ii) Each nilpotent element a of a t -norm T is also a zero divisor of T (if $n > 1$ is the smallest integer such that $a_T^{(n)} = 0$ then $T(a, a_T^{(n-1)}) = 0$ with $a_T^{(n-1)} > 0$), but not conversely (T^{nM} is a counterexample).

- (iii) If a t -norm T has a nilpotent element a then there is always an element $b \in]0, 1[$ such that $b_T^{(2)} = 0$. Indeed, if $n > 1$ is the smallest integer such that $a_T^{(n)} = 0$ then $b = a_T^{(n-1)}$ satisfies $b_T^{(2)} = 0$.
- (iv) If $a \in]0, 1[$ is a nilpotent element of a t -norm T then each number $b \in]0, a[$ is also a nilpotent element of T , i.e., the set of nilpotent elements of a t -norm T can either be the empty set (as for T_M or T_P) or an interval of the form $]0, c[$ or $]0, c]$. The same is true for zero divisors.

Definition 2.8.

- (i) A t -norm T is called *strict* if it is continuous and strictly monotone,
- (ii) A t -norm T is called *nilpotent* if it is continuous and if each $a \in]0, 1[$ is a nilpotent element of T .

Example 2.7.

- The product T_P is a strict t -norm, and the Lukasiewicz t -norm T_L is a nilpotent t -norm. so each strict t -norm is isomorphic to T_P and each nilpotent t -norm is isomorphic to T_L .

Proposition 2.4.

- (i) An element $a \in [0, 1]$ is an idempotent element of a t -norm T if and only if for all $x \in [a, 1]$ we have $T(a, x) = \min(a, x)$.
- (ii) If T is a continuous t -norm, then $a \in [0, 1]$ is an idempotent element of T if and only if for all $x \in [0, 1]$ we have $T(a, x) = \min(a, x)$.

Proof.

If for all $x \in [a, 1]$ we have $T(a, x) = \min(a, x)$ then also $T(a, a) = a$. The converse in (i) is an immediate consequence of the monotonicity (T3) and the boundary condition (T4) of T . When proving (ii), because of (i) it suffices to show that for an idempotent element $a \in [0, 1]$ we have $T(a, x) = x$ for all $x \in [0, a]$. Since the function $T(a, \cdot) : [0, a] \rightarrow [0, a]$ satisfies $T(a, 0) = 0$ and $T(a, a) = a$, the continuity of T implies that there is some $z \in [0, a]$ with $T(a, z) = x$, leading to $T(a, x) = T(a, T(a, z)) = T(T(a, a), z) = T(a, z) = x = \min(a, x)$,

which completes the proof. $\square$

Definition 2.9. [6]

A T -norm T is said Archimedean if :

(i) T is continue.

(ii) $T(x, x) < x, \forall x \in [0, 1]$.

Definition 2.10. [6]

A T -norm Archimedean T is said strict if:

$T(x', y') < T(x, y)$ if $x' < x$ and $y' < y \forall x, x', y, y' \in [0, 1]$.

Example 2.8.

The Łukasiewicz t -norm T_L and product t -norm T_P are Archimedean.

Definition 2.11. [9]

For an arbitrary t -norm T we consider the following properties:

(i) The t -norm T is said to be strictly monotone if :

$$T(x, y) < T(x, z) \text{ whenever } x > 0 \text{ and } y < z. \quad (\text{SM})$$

(ii) The t -norm T satisfies the cancellation law if :

$$T(x, y) = T(x, z) \text{ implies } x = 0 \text{ or } y = z. \quad (\text{CL})$$

(iii) The t -norm T satisfies the conditional cancellation law if :

$$T(x, y) = T(x, z) > 0 \text{ implies } y = z. \quad (\text{CCL})$$

(iv) The t -norm T has the limit property if :

$$\text{for all } (x, y) \in]0, 1[: \lim_{n \rightarrow \infty} x_T^{(n)} = 0. \quad (\text{LP})$$

Example 2.9.

- (i) The minimum T_M has none of these properties, and the product T_P satisfies all of them.
- (ii) The Lukasiewicz t -norm T_L and the drastic product T_D are Archimedean and satisfy the conditional cancellation law (CCL) and the limit property (LP), but none of the other properties.
- (iii) The t -norm T given by:

$$T(x, y) = \begin{cases} \frac{xy}{2} & \text{if } (x, y) \in [0, 1]^2 \\ \min(x, y) & \text{otherwise} \end{cases}.$$

which is strictly monotone and satisfies the cancellation law (CL).

Proposition 2.5.

Let T be a t -norm. Then we have:

- (i) T is strictly monotone if and only if it satisfies the cancellation law (CL).
- (ii) If T is strictly monotone then it has only trivial idempotent elements.
- (iii) If T is strictly monotone then it has no zero divisors.

Proof.

Obviously, the strict monotonicity (SM) of T implies the validity of the cancellation law (CL). Conversely, the strict monotonicity follows from the cancellation law together with the monotonicity (T3), thus proving (i). Assertion (ii) follows immediately from $T(x, x) < T(x, 1) = x$ for all $x \in]0, 1[$. Concerning (iii), the assumption that $a \in]0, 1[$ is a zero divisor, i.e., $T(a, b) = 0$ for some $b \in]0, 1[$, leads to $T(a, \frac{b}{2}) = T(a, b) = 0$ and, therefore, violates the strict monotonicity of T . $\square$

2.1.3 Additive and multiplicative generators**Definition 2.12.** [9]

An additive generator $t : [0, 1] \rightarrow [0, \infty]$ of a t -norm T is a strictly decreasing function which is also right-continuous in 0 and satisfies $t(1) = 0$, such that for all $(x, y) \in [0, 1]^2$ we have:

$$T(x, y) = t^{(-1)}(t(x) + t(y)).$$

Remark 2.5.

If $t : [0, 1] \rightarrow [0, \infty]$ is an additive generator of some t -norm T , then we clearly have for all $x_1, x_2, \dots, x_n \in [0, 1]$

$$T(x_1, x_2, \dots, x_n) = t^{(-1)}\left(\sum_{i=1}^n t(x_i)\right)$$

Example 2.10.

- If we have the function $t : [0, 1] \rightarrow [0, \infty]$ given by : $t(x) = 1 - x$, we get the Lukasiewicz t -norm T_L .

- If we have the function $t(x) = -\ln x$, we get the product T_P .

- For the drastic product T_D (which is right-continuous but not continuous) is obtained putting

$$t(x) = \begin{cases} 2 - x & \text{if } x \in [0, 1[\\ 0 & \text{if } x = 1 \end{cases}$$

Definition 2.13.

A multiplicative generator of a t -norm T is a strictly increasing function $\theta : [0, 1] \rightarrow [0, 1]$ which is right-continuous in 0 and satisfies $\theta(1) = 1$, such that for all $(x, y) \in [0, 1]^2$ we have:

$$T(x, y) = \theta^{(-1)}(\theta(x) \theta(y)).$$

Remark 2.6.

- If T is a continuous Archimedean t -norm and if $t : [0, 1] \rightarrow [0, \infty]$, is an additive generator of the t -norm T , so the function $\theta : [0, 1] \rightarrow [0, 1]$ given by : $\theta(x) = \exp(-t(x))$ is a multiplicative generator of T .

2.2 Triangular conorms

In this section, we recall some basic definitions and notions concerning triangular conorms.

2.2.1 Basic definitions and properties

Definition 2.14. [10] A triangular conorm (t -conorm for short) is a binary operation S on the unit interval $[0, 1]$, i.e., it is a function $S : [0, 1]^2 \rightarrow [0, 1]$ such that for all $x, y, z \in [0, 1]$: the following four axioms are satisfied:

(S1) $S(x, y) = S(y, x)$. (commutativity)

(S2) $S(x, S(y, z)) = S(S(x, y), z)$. (associativity)

(S3) $S(x, y) \leq S(x, z)$ whenever $y \leq z$ (monotonicity)

(S4) $S(x, 0) = x$. (boundary condition)

Example 2.11.

The following are the four basic t -conorms S_M , S_P , S_L , and S_D given by, respectively:

$S_M(x, y) = \max(x, y)$	(maximum) Figure 2.5
$S_P(x, y) = x + y - x \cdot y$	(probabilistic sum) Figure 2.6
$S_L(x, y) = \min(x + y, 1)$	(Łukasiewicz t -conorm, bounded sum) Figure 2.7
$S_D(x, y) = \begin{cases} 1 & \text{if } (x, y) \in]0, 1]^2 \\ \max(x, y) & \text{otherwise} \end{cases}$	(drastic sum) Figure 2.8

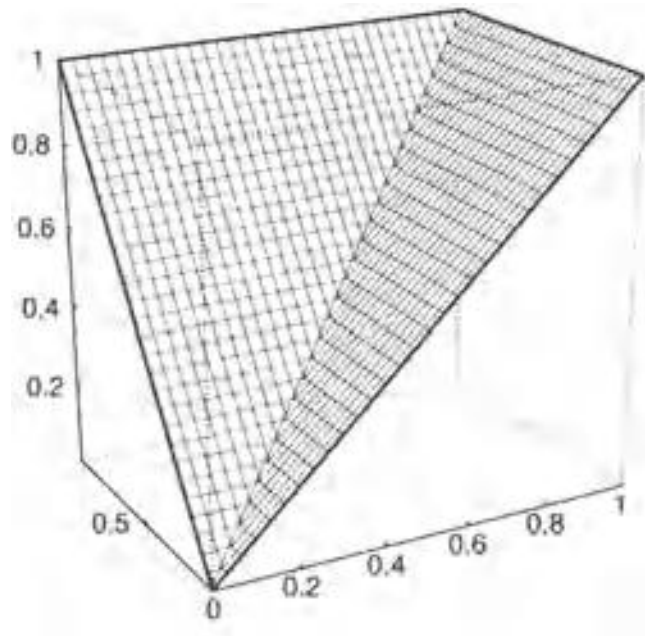


Figure 2.5: Maximum t -conorm

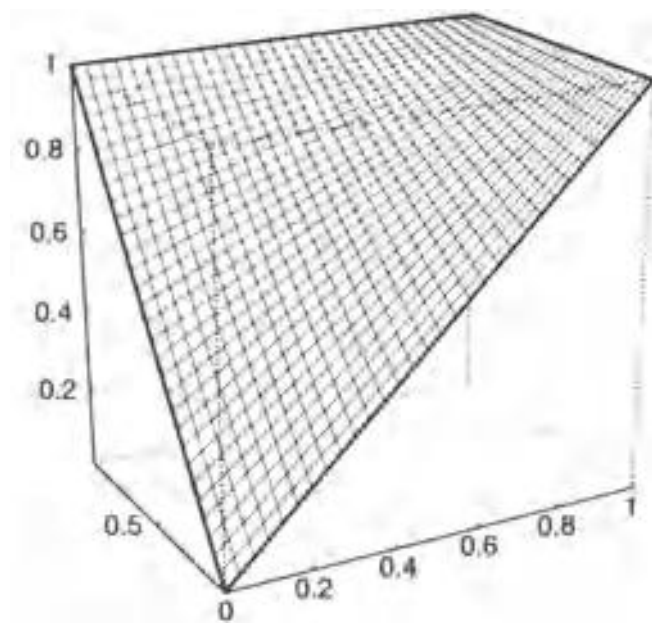


Figure 2.6: Probabilistic Sum t-conorm

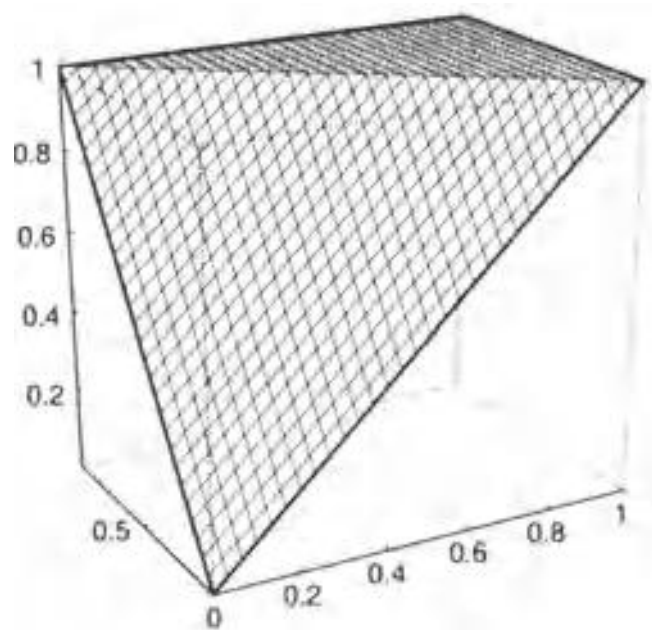


Figure 2.7: Łukasiewicz t-conorm

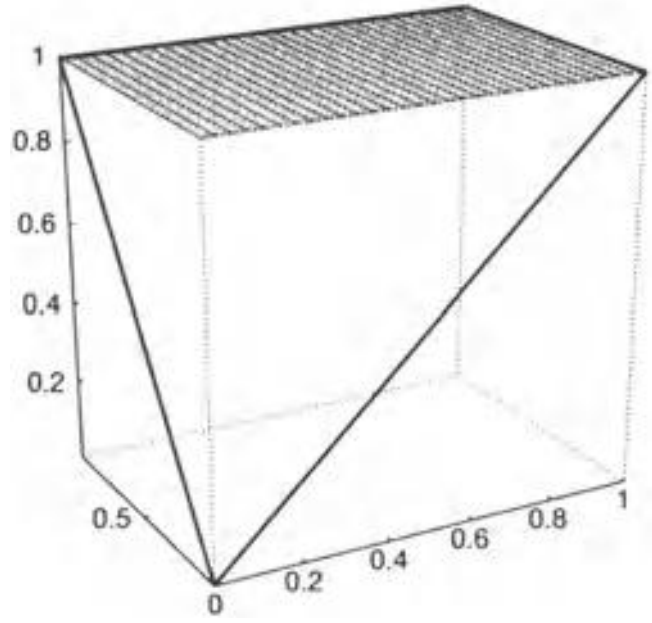


Figure 2.8: Drastic Sum t-conorm

Example 2.12.

$T(x, y) = \frac{x + y}{(1 + xy)}$	<i>Einstein</i>
$T(x, y) = \frac{x + y - 2xy}{(1 - xy)}$	<i>Hamacher</i>
$T(x, y) = \frac{x + y + xy - \min(x, y, 1 - \alpha)}{\max(1 - \alpha, 1 - y, \alpha)}$	<i>Dubois and Parade (1986) $\alpha \in [0, 1]$</i>

Proposition 2.6.

Any t-conorm S satisfies $S(1, x) = S(x, 1) = 1$, for all $x \in [0, 1]$.

Proof.

We know that $S(x, 1) \in [0, 1]$, so $S(x, 1) \leq 1$, and we use the axiom (S3)(monotonicity), we obtient $S(x, 1) \geq S(0, 1) = 1$. □

Proposition 2.7. [10]

A function $S : [0, 1]^2 \rightarrow [0, 1]$ is a t -conorm if and only if there exists a t -norm T such that for all $(x, y) \in [0, 1]^2$

$$S(x, y) = 1 - T(1 - x, 1 - y). \quad (*)$$

Proof.

If T is a t -norm then obviously the operation S defined by $(*)$ satisfies (S1)- (S3) and (S4)

(S₁)

$$\begin{aligned} S(x, y) &= 1 - T(1 - x, 1 - y) \\ &= 1 - (1 - y, 1 - x) \\ &= S(y, x); \end{aligned}$$

(S₂)

$$\begin{aligned} S(x, S(y, z)) &= 1 - T(1 - x, 1 - S(y, z)) \\ &= 1 - T(1 - x, 1 - (1 - T(1 - y, 1 - z))) \\ &= 1 - T(1 - x, T(1 - y, 1 - z)); \end{aligned} \quad (2.1)$$

$$\begin{aligned} S(S(x, y), z) &= 1 - T(1 - S(x, y), 1 - z) \\ &= 1 - T(1 - (1 - T(1 - x, 1 - y)), 1 - z) \\ &= 1 - T(T(1 - x, 1 - y), 1 - z) \\ &= 1 - T(1 - x, T(1 - y, 1 - z)); \end{aligned} \quad (2.2)$$

(S₃) $S(x, y) = 1 - T(1 - x, 1 - y) \leq 1 - T(1 - x, 1 - z) = S(x, z)$ whenever $y \leq z$,

(S₄) $S(x, 0) = 1 - T(1 - x, 1) = 1 - (1 - x) = x$,

and is, therefore, a t -conorm. On the other hand, if S is a t -conorm, then define the function $T : [0, 1]^2 \rightarrow [0, 1]$ by

$$T(x, y) = 1 - S(1 - x, 1 - y), \quad (**)$$

Again, it is trivial to T is a t -norm and that $(*)$ holds. $\square$

Remark 2.7.

(i) The t -conorm given by $(*)$ is called the dual t -conorm of T and, analogously, the t -norm given by $(**)$ is said to be the dual t -norm of S .

(ii) The proof of **Proposition 2.7** makes it clear that also each t -norm is the dual operation of some t -conorm. Note that (T_M, S_M) , (T_P, S_P) , (T_L, S_L) , and (T_D, S_D) are pairs of t -norms and t -conorms which are mutually dual to each other.

Definition 2.15. [9]

Let T be a t -norm and S be a t -conorm. Then we say that T is distributive over S if for all $x, y, z \in [0, 1]$

$$T(x, S(y, z)) = S(T(x, y), T(x, z)).$$

and that S is distributive over T if for all $x, y, z \in [0, 1]$

$$S(x, T(y, z)) = T(S(x, y), S(x, z)).$$

Remark 2.8.

If T is distributive over S and S is distributive over T , then (T, S) is called a distributive pair (of t -norms and t -conorms).

Proposition 2.8. [9]

Let T be a t -norm and S a t -conorm. Then we have:

- (i) S is distributive over T if and only if $T = T_M$.
- (ii) T is distributive over S if and only if $S = S_M$.
- (iii) (T, S) is a distributive pair if and only if $T = T_M$ and $S = S_M$.

Remark 2.9.

- (i) The duality changes the order: if, for some t -norms T_1 and T_2 we have $T_1 \leq T_2$, and if S_1 and S_2 are the dual t -conorms of T_1 and T_2 , respectively, then we get $S_1 \geq S_2$. Consequently, for each t -conorm S , we have

$$S_M \leq S \leq S_D \tag{2.3}$$

i.e., the maximum S_M is the weakest and the drastic sum S_D is the strongest t -conorm.

- (ii) For the t -conorms in **example 2.11** we get this ordering:

$$S_M < S_P < S_L < S_D. \tag{2.4}$$

The continuity of t-conorm S is equivalent to the continuity of the t-norm dual T .

Definition 2.16.

A T -conorm $S : [0, 1]^2 \rightarrow [0, 1]$ is continue if of for all convergent sequences $(x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}} \in [0, 1]^{\mathbb{N}}$ we have :

$$S \left(\lim_{n \rightarrow \infty} x_n, \lim_{n \rightarrow \infty} y_n \right) = \lim_{n \rightarrow \infty} S(x_n, y_n).$$

Example 2.13.

- the t-conorms S_M, S_P, S_L are continues, and the drastic sum S_D is not continue.

2.2.2 Elementary algebraic properties

Definition 2.17. [9]

(i) An element $a \in [0, 1]$ is called an idempotent element of S if $S(a, a) = a$.

The numbers 0 and 1 (which are idempotent elements for each t-conorm S) are called trivial idempotent elements of S , each idempotent element in $]0, 1[$ will be called a non-trivial idempotent element of S .

(ii) An element $a \in]0, 1[$ is called a nilpotent element of S if there exists some $n \in \mathbb{N}$ such that $a_S^{(n)} = 0$.

(iii) An element $a \in]0, 1[$ is called a zero divisor of S if there exists some $b \in]0, 1[$ such that $S(a, b) = 0$.

Definition 2.18.

For an arbitrary t-conorm S we consider the following properties:

(i) The t-conorm S is said to be strictly monotone if :

$$S(x, y) < S(x, z) \text{ whenever } x < 1 \text{ and } y < z. \quad (\text{SM})$$

(ii) The t-conorm s satisfies the cancellation law if :

$$S(x, y) = S(x, z) \text{ implies } x = 0 \text{ or } y = z. \quad (\text{CL})$$

(iii) The t -conorm S satisfies the conditional cancellation law if :

$$S(x, y) = S(x, z) > 0 \text{ implies } y = z. \quad (\text{CCL})$$

(iv) The t -conorm S has the limit property if :

$$\text{for all } (x, y) \in]0, 1[: \lim_{n \rightarrow \infty} x_S^{(n)} = 0. \quad (\text{LP})$$

Definition 2.19.

A t -conorm S is said Archimedean if :

(i) S is continue.

(ii) $S(x, x) > x, \forall x \in [0, 1]$.

2.2.3 Additive and multiplicative generators

Because of the duality between t -norms and t -conorms, telling that T is a t -norm if and only if the function $S : [0, 1]^2 \rightarrow [0, 1]$ given by:

$$S(x, y) = n(T(n(x), n(y))).$$

is a t -conorm, additive and multiplicative generators of t -conorms can also be considered like that:

Let T be a t -norm, S the dual t -conorm, $t : [0, 1] \rightarrow [0, \infty]$ an additive generator of T and $\theta : [0, 1] \rightarrow [0, 1]$ a multiplicative generator of T .

Define the functions $\varphi : [0, 1] \rightarrow [0, \infty]$ and $\xi : [0, 1] \rightarrow [0, 1]$ by :

$$\varphi(x) = t(1 - x).$$

$$\xi(x) = \theta(1 - x).$$

then it is obvious that for all $(x, y) \in [0, 1]^2$, we get :

$$S(x, y) = \varphi^{-1}(\varphi(x) + \varphi(y)).$$

$$S(x, y) = \xi^{(-1)}(\xi(x) \cdot \xi(y)).$$

Remark 2.10.

• If S is Archimedean and continuous t -conorm so the dual t -norm is Archimedean and continuous and, consequently, an additive generator $t : [0, 1] \rightarrow [0, \infty]$ and $\varphi : [0, 1] \rightarrow [0, \infty]$ given by $\varphi(x) = t(1 - x)$ is an additive generator of S , and $\xi : [0, 1] \rightarrow [0, 1]$ given by $\xi(x) = \exp(-t(1 - x))$ is a multiplicative generator of S .

Chapter 3

A t-partial order obtained from t-norms

3.1 Triangular order

A triangular order (t-order for short) is a partial order on a bounded lattice. Obtained from a t-norm on this bounded lattice.

Definition 3.1. [8] *Let L be a bounded lattice, let T be a t-norm on L . The order defined as follows is called a t-order (triangular order) for a t-norm T .*

$$x \preceq_T y :\Leftrightarrow T(l, y) = x \text{ for some } l \in L.$$

Example 3.1.

Let $L = \{0, a, b, c, 1\}$ and consider the order $\leq$ on L as in **Figure 3.1**. We choose the t-norm T_D . Then $a \leq b$, but $a \not\preceq_{T_D} b$. We suppose that $a \preceq_{T_D} b$. Then there exists an element $l \in L$ such that $T_D(l, b) = a$. If $l = 0$, then it is obtained that $a = 0$, a contradiction. If $l = a, b$ or c , then we have that $T_D(l, b) = 0 = a$, a contradiction. If $l = 1$, then it is obtained that $T_D(1, b) = b = a$, which is not possible. So, there does not exist any element $l \in L$ satisfying $T_D(l, b) = a$. Thus, $a \not\preceq_{T_D} b$. Then the order $\preceq_{T_D}$ on L is as in **Figure 3.2**

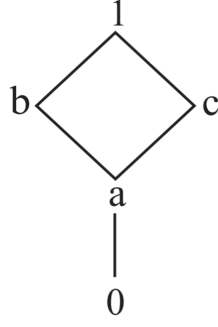


Figure 3.1: The order $\leq$ on L.

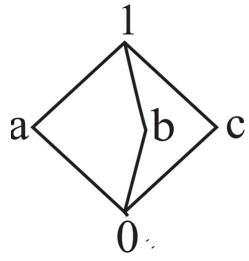


Figure 3.2: The order $\preceq_D$ on L.

Proposition 3.1.

Let L be a bounded lattice, let T be a t -norm on L . Then the binary relation $\preceq_T$ is a partial order on L .

Proof.

Since $1 \in L$ and $T(1, x) = x$, $x \preceq_T x$ holds. Thus, the reflexivity is satisfied.

Let $x \preceq_T y$ and $y \preceq_T x$. Then, there exist l_1, l_2 of L such that $T(l_1, y) = x$ and $T(l_2, x) = y$. Hence, $x = T(l_1, y) \leq T(1, y) = y$; i.e., $x \leq y$. On the other hand, $y = T(l_2, x) \leq T(1, x) = x$; i.e., $y \leq x$. So, $x = y$. Thus, the antisymmetry is satisfied.

Let $x \preceq_T y$ and $y \preceq_T z$. Then, there exist l_1, l_2 of L such that $T(l_1, y) = x$ and $T(l_2, z) = y$. For $T(l_1, l_2)$ of L , $T(T(l_1, l_2), z) = T(l_1, T(l_2, z)) = T(l_1, y) = x$. Thus, $x \preceq_T z$. This means that the relation $\preceq_T$ satisfies the transitivity. So, we have that $\preceq_T$ is a partial order on L . $\square$

Definition 3.2. This partial order $\preceq_T$ is called a T -partial order on L .

Proposition 3.2.

The T -partial order obtained from a triangular norm T is included in $\leq$.

Proof.

Let $(x, y) \in \preceq_T$. Then, there exists an element l of L such that $x = T(l, y) \leq T(1, y) = y$. Thus $(x, y) \in \leq$. $\square$

Remark 3.1.

*The converse of the **Proposition 3.2**, is in general not true (**example 3.1**).*

Proposition 3.3.

Let L be a bounded lattice and T be a t -norm on L . We can show that $\preceq_T$ is equal to $\leq$ if and only if T is a divisible t -norm on L .

Proof.

We always, $\preceq_T \subseteq \leq$.

So, it suffices to show that $\leq \subseteq \preceq_T \Leftrightarrow \forall x, y \in L$ such as $x \leq y, \exists z \in L : x = T(y, z)$.

($\Rightarrow$) let $x, y \in L$ such as $x \leq y$ according to the proposition $\exists z \in L$ such as $x = T(y, z)$, So, T is divisible.

($\Leftarrow$) let $x, y \in L$ such as $x \leq y$ according to the proposition $\exists z \in L$ such as $x = T(y, z)$, so $x \preceq_T y$.

$\square$

3.2 Some properties of the partially ordered set $(L, \preceq_T)$

In this Section, we show that $(L, \preceq_T)$ is a partially ordered set. In this section, we give some examples for t -norms such that $(L, \preceq_T)$ is a lattice or not.

Remark 3.2. [7]

Even if $(L, \leq)$ is a chain, $(L, \preceq_T)$ may not be a chain. For example, consider the lattice $L = [0, 1]$ and the nilpotent minimum t -norm T^{nM} defined by

$$T^{nM}(x, y) = \begin{cases} 0 & \text{if } x + y \leq 1, \\ \min(x, y) & \text{otherwise,} \end{cases}$$

$\frac{1}{3} \leq \frac{1}{2}$, but $\frac{1}{2}$ and $\frac{1}{3}$ aren't comparable with respect to the relation T^{nM} on $[0, 1]$. Indeed; if $\frac{1}{2} \preceq_{T^{nM}} \frac{1}{3}$, by **Proposition 3.2**, then $\frac{1}{2} \leq \frac{1}{3}$ which is a contradiction. Therefore, $\frac{1}{2} \not\preceq_{T^{nM}} \frac{1}{3}$. On the other hand, if $\frac{1}{3} \preceq_{T^{nM}} \frac{1}{2}$, then there exists an element $l \in [0, 1]$ such that $T^{nM}(l, \frac{1}{2}) = \frac{1}{3}$. Since $T^{nM}(l, \frac{1}{2}) = \frac{1}{3}$, $l + \frac{1}{2} > 1$; i.e., $l > \frac{1}{2}$. Thus, $T^{nM}(l, \frac{1}{2}) = \min(l, \frac{1}{2}) = \frac{1}{2}$, which is impossible. Therefore, $\frac{1}{3} \not\preceq_{T^{nM}} \frac{1}{2}$ as required. Hence, $\frac{1}{2}$ isn't comparable with $\frac{1}{3}$ according to the relation $\preceq_{T^{nM}}$ on $[0, 1]$

Remark 3.3.

Let L be a bounded lattice. Consider a t -norm T on L . For $X \subseteq L$, we denote the set of the upper bounds of X with respect to $\preceq_T$ on L by $\overline{X}_T$. Also, we denote the set of the lower bounds of X with respect to $\preceq_T$ on L by $\underline{X}_T$. Also, 0 is the smallest element and 1 is the greatest element with respect to $\preceq_T$. If there exists the greatest element of the lower bounds and the least element of the upper bounds with respect to $\preceq_T$, respectively, we will denote by $\bigwedge_T$ and $\bigvee_T$.

Example 3.2.

Let $X = \{a, b\}$, since $T(a, b) \preceq_T a$ and $T(a, b) \preceq_T b$, we have that $T(a, b) \in \underline{\{a, b\}}_T$. Thus, $\underline{\{a, b\}}_T \neq \emptyset$. Since $T(a, 1) = a$ and $T(b, 1) = b$, $a \preceq_T 1$ and $b \preceq_T 1$. Thus, $1 \in \overline{\{a, b\}}_T$, so we obtain that $\overline{\{a, b\}}_T \neq \emptyset$.

Remark 3.4.

L may not be a lattice with respect to the order $\preceq_T$. The following example illustrates that.

Example 3.3.

Let $L = [0, 1]$ and T^{nM} be the nilpotent minimum t -norm on $[0, 1]$. Then, $(L, \preceq_{T^{nM}})$ is a meet-semilattice, but not a join-semilattice.

If $x \preceq_{T^{nM}} y$ or $y \preceq_{T^{nM}} x$, then $x \wedge_{T^{nM}} y$ is equal to x or y . Let $x \not\preceq_{T^{nM}} y$ and $y \not\preceq_{T^{nM}} x$. It must be $x + y \leq 1$. Otherwise, if $x + y > 1$, then $T^{nM}(x, y) = \min(x, y)$. Thus, if $T^{nM}(x, y) = x$ or y , then we have that $x \preceq_{T^{nM}} y$ or $y \preceq_{T^{nM}} x$, which are contradictions.

Let $x + y \leq 1$. It is clear that $0 \in \underline{\{x, y\}}_{T^{nM}}$. We claim that $x \wedge_{T^{nM}} y = 0$. Let $k \in \underline{\{x, y\}}_{T^{nM}} \setminus \{0\}$. Then, $k \preceq_{T^{nM}} x$ and $k \preceq_{T^{nM}} y$. There exist two elements l_1, l_2 of $[0, 1]$ such

that $k = T^{nM}(x, l_1) = T^{nM}(y, l_2)$. Since $0 \neq k = T^{nM}(x, l_1) = T^{nM}(y, l_2)$ and $T^{nM}(x, l_1) \neq 0$, we have that $x + l_1 > 1$ and $T^{nM}(x, l_1) = \min(x, l_1)$. If $T^{nM}(x, l_1) = \min(x, l_1) = x$, then $k = x \preceq_{T^{nM}} y$. Since x and y aren't comparable, this is a contradiction. Then, we have that $T^{nM}(x, l_1) = l_1 = k$. Since $x + l_1 > 1$ and $x + y \leq 1$, we obtain that $k = l_1 > 1 - x \geq y$. Since $0 \neq k = T^{nM}(y, l_2)$, $k = T^{nM}(y, l_2) = \min(y, l_2)$. If $k = T^{nM}(y, l_2) = \min(y, l_2) = y$, then this contradicts the fact that x and y aren't comparable. Then $k = T^{nM}(y, l_2) = l_2$. Since $k = l_1 = l_2 > 1 - x \geq y$, $T^{nM}(y, k) = \min(y, k) = y$. This is a contradiction since x and y aren't comparable. So, $k = 0$. Hence, $x \wedge_{T^{nM}} y$ exists.

Now, let us show that $(L, \preceq_{T^{nM}})$ is not a join-semilattice. By **Remark 3.2**, we know that $\frac{1}{2}$ and $\frac{1}{3}$ aren't comparable. Let $k \in \overline{\{\frac{1}{2}, \frac{1}{3}\}}_{T^{nM}}$. Then, $\frac{1}{2} \preceq_{T^{nM}} k$ and $\frac{1}{3} \preceq_{T^{nM}} k$. Thus, there exist l_1 and l_2 such that $\frac{1}{2} = T^{nM}(k, l_1)$ and $\frac{1}{3} = T^{nM}(k, l_2)$. It follows from $\frac{1}{2} = \min(k, l_1)$ and $\frac{1}{3} = \min(k, l_2)$ that $k + l_1 > 1$ and $k + l_2 > 1$. By **Remark 3.2**, we have that $l_1 = \frac{1}{2}$ and $l_2 = \frac{1}{3}$. Therefore, since $\frac{1}{2} = T^{nM}(k, \frac{1}{2})$ and $\frac{1}{3} = T^{nM}(k, \frac{1}{3})$, $k + \frac{1}{2} > 1$ and $k + \frac{1}{3} > 1$. Hence, we have that $k > \frac{2}{3} > \frac{1}{2}$, and so $\overline{\{\frac{1}{2}, \frac{1}{3}\}}_{T^{nM}} \subseteq]\frac{2}{3}, 1]$.

Now, we want to show that $\overline{\{\frac{1}{2}, \frac{1}{3}\}}_{T^{nM}} =]\frac{2}{3}, 1]$. If $x \in]\frac{2}{3}, 1]$, then $x > \frac{2}{3}$. Now, let us prove that $x \succeq_{T^{nM}} \frac{1}{2}$ and $x \succeq_{T^{nM}} \frac{1}{3}$. We investigate whether there exist two elements x_1 and x_2 such that $\frac{1}{2} = T^{nM}(x, x_1)$ and $\frac{1}{3} = T^{nM}(x, x_2)$. If we choose $x_1 = \frac{1}{2}$ and $x_2 = \frac{1}{3}$, then we have that $\frac{1}{2} = T^{nM}(x, \frac{1}{2})$ and $\frac{1}{3} = T^{nM}(x, \frac{1}{3})$. Therefore, we obtain that $x \succeq_{T^{nM}} \frac{1}{2}$ and $x \succeq_{T^{nM}} \frac{1}{3}$. Since there doesn't exist the least element of $\overline{\{\frac{1}{2}, \frac{1}{3}\}}_{T^{nM}} =]\frac{2}{3}, 1]$ with respect to $\preceq_{T^{nM}}$, $([0, 1], \preceq_{T^{nM}})$ is not a join-semilattice.

Proposition 3.4.

Let L be a bounded lattice and T be a t -norm on L . If $T = T_D$, then for arbitrary $a \in L \setminus \{0, 1\}$ it holds that $a \wedge_T b = 0$ and $a \vee_T b = 1$ for every $b \in L \setminus \{0, 1, a\}$. Thus, $(L, \preceq_{T_D})$ is a lattice.

Proof.

For every $a, b \neq 0, 1$ and $a \neq b$, since $T_D(a, b) = 0$ and for all $k \in L$, $T_D(a, k) \neq b$ and $T_D(b, k) \neq a$, a and b are not comparable with respect to $\preceq_{T_D}$. We claim that for arbitrary $a \in L \setminus \{0, 1\}$ it satisfies $a \wedge_{T_D} b = 0$ for every $b \in L \setminus \{0, 1, a\}$. If $a \wedge_{T_D} b = x \neq 0$, then $x \preceq_{T_D} a$ and $x \preceq_{T_D} b$. Thus, there exists $x_1 \in L \setminus \{0\}$ such that $0 \neq x = T_D(a, x_1)$. If $x = a$, then this is a contradiction since a and b aren't comparable with respect to $\preceq_{T_D}$. If $T_D(a, x_1) = 1$ or x_1 , then we obtain that $a = 1$. This contradicts to the choice of a . Hence, $a \wedge_{T_D} b = 0$. Similarly, let us show that for arbitrary $a \in L \setminus \{0, 1\}$, $a \vee_{T_D} b = 1$ for every $b \in L \setminus \{0, 1, a\}$.

Let $a \vee_{T_D} b = x$. Then $a \preceq_{T_D} x$ and $b \preceq_{T_D} x$, and so there exist $x_1, x_2 \in L \setminus \{0\}$ such that $T_D(x, x_1) = a$ and $T_D(x, x_2) = b$. If $x = a$, then $T_D(a, x_2) = b$ which is a contradiction since a and b aren't comparable with respect to $\preceq_{T_D}$. Then, $x_1 = a$, so it must be $x = 1$. Therefore, $a \vee_{T_D} b = 1$. Finally, we have that $(L, \preceq_{T_D})$ is a lattice.

Now, we give an example such that $(L, \preceq_T)$ is a lattice and $T \neq T_D$. $\square$

Example 3.4.

Consider the t-norm

$$T(x, y) = \begin{cases} 0 & \text{if } (x, y) \in]0, \frac{1}{2}[^2, \\ \min(x, y) & \text{otherwise,} \end{cases}$$

on $[0, 1]$ ([16], p. 18, 1.24 Example). Then $([0, 1], \preceq_T)$ is a lattice.

Choose that x and y aren't comparable with respect to $\preceq_T$. Otherwise, it is trivial. We want to show that $x \vee_T y = \frac{1}{2}$. If $x \notin]0, \frac{1}{2}[$ or $y \notin]0, \frac{1}{2}[$, then $T(x, y) = \min(x, y)$. This contradicts that x and y aren't comparable with respect to $\preceq_T$. Therefore, $x, y \in]0, \frac{1}{2}[$.

Now, let us show that $\frac{1}{2} \in \overline{\{x, y\}_T}$. From $T(x, \frac{1}{2}) = \min(x, \frac{1}{2}) = x$, we have that $x \preceq_T \frac{1}{2}$. Similarly, $y \preceq_T \frac{1}{2}$ holds. Therefore, we obtain that $\frac{1}{2} \in \overline{\{x, y\}_T}$. Choose arbitrary $k \in \overline{\{x, y\}_T}$. Then, $x \preceq_T k$ and $y \preceq_T k$. There exist $l_1, l_2 \in [0, 1]$ satisfying that $x = T(k, l_1)$ and $y = T(k, l_2)$. Then, $0 \neq x = T(k, l_1) = \min(k, l_1)$. Since x and y aren't comparable, the case $x = k$ is not possible. Then, $l_1 = x \in]0, \frac{1}{2}[$. Hence, $k \notin]0, \frac{1}{2}[$ and $k \neq 0$; i.e., $k \geq \frac{1}{2}$. Since $\frac{1}{2} = T(k, \frac{1}{2}) = \min(k, \frac{1}{2})$, we have that $k \succeq_T \frac{1}{2}$, and so $x \vee_T y = \frac{1}{2}$. Choose arbitrary $k \in \overline{\{x, y\}_T}$. Since $k \preceq_T x$ and $k \preceq_T y$, there exist $x_1, y_1 \in [0, 1]$ such that $k = T(x, x_1)$ and $k = T(y, y_1)$. Since x and y aren't comparable with respect to $\preceq_T$, it is not possible $k = x$ or $k = y$. Therefore, $k = x_1$ and $k = y_1$. It follows from $k = T(x, k)$ that $k \leq x < \frac{1}{2}$. Therefore, $x, k \in]0, \frac{1}{2}[$ implies $k = T(x, k) = 0$. So, $x \wedge_T y = 0$. Hence, $([0, 1], \preceq_T)$ is a lattice.

Proposition 3.5. [7]

Let L be a lattice and T be any t-norm on L . If $a \preceq_T b$ for $a, b \in L$, then $T(a, c) \preceq_T T(b, c)$ for every $c \in L$.

Proof.

Let $a \preceq_T b$ for $a, b \in L$. Then, there exists $x \in L$ such that $T(x, b) = a$. Since $T(a, c) = T(T(x, b), c) = T(x, T(b, c))$, $T(a, c) \preceq_T T(b, c)$. So, this shows that the monotonicity holds. $\square$

Corollary 3.1.

Let L be a lattice and T be any t -norm on L . If $(L, \preceq_T)$ is a lattice, then $T : (L, \preceq_T)^2 \rightarrow (L, \preceq_T)$ is a t -norm.

Proposition 3.6.

Let $(L, \leq)$ be a bounded lattice and T be a t -norm on L . If $(L, \preceq_T)$ is a chain, then T is a divisible t -norm; i.e., $\leq = \preceq_T$.

Proof.

For $a, b \in L$, let $a < b$ and **Proposition 3.2** $a \not\prec_T b$. Since $(L, \preceq_T)$ is a chain, $b \prec_T a$, and so $b < a$ by . This is a contradiction. Therefore, $\preceq_T = \leq$. $\square$

Notation 3.1. We note by H_T the set of all idempotent elements of t -norm T .

Also, we note by A the subset of all elements of L which are the supremum of a subset of atoms.

3.3 Some determinations on sets H_T and A

Proposition 3.7.

Let $L = [0, 1]$ and T be any t -norm on L . Then, $(H_T, \preceq_T)$ is a chain.

Proof.

Let a and b be two idempotent elements on $[0, 1]$. Since a and b are two elements of $[0, 1]$, we have that $a \leq b$ or $b \leq a$. Suppose that $a \leq b$. Since $a = T(a, a) \leq T(a, b) \leq T(a, 1) = a$, $T(a, b) = a$. So, $a \preceq_T b$. If $b \leq a$, then similarly we obtain that $b \preceq_T a$. Hence, $(H_T, \preceq_T)$ is a chain. $\square$

Theorem 3.1. [7]

Let L be a complete lattice and T be any t -norm on L . Then $a \wedge_T b = T(a, b)$ for every $a, b \in H_T$.

Proof.

Since $T(a, b) \preceq_T a$ and $T(a, b) \preceq_T b$, $T(a, b) \in \underline{\{a, b\}}_T$. Let $l \in \underline{\{a, b\}}_T$ be arbitrary. This implies that $l \preceq_T a$ and $l \preceq_T b$. In that case, there exist two elements $a_1, b_1 \in L$ such that $l = T(a, a_1) = T(b, b_1)$. Also, $T(l, b) = T(T(b, b_1), b) = T(T(b, b), b_1) = T(b, b_1) = l$ and

similarly $T(l, a) = T(T(a, a_1), a) = T(T(a, a), a_1) = T(a, a_1) = l$ holds. Since $l = T(l, a) \preceq_T T(b, a) = T(a, b)$ by monotonicity of T on $(L, \preceq_T)$, we have that $T(a, b)$ is the greatest element of the lower bounds of $\{a, b\}$ with respect to $\preceq_T$ and so $T(a, b) = a \wedge_T b$. $\square$

Definition 3.3. [7]

(i) A t -norm T on a lattice L is called $\vee$ -distributive if

$$T(a, b_1 \vee b_2) = T(a, b_1) \vee T(a, b_2) \quad (3.1)$$

for every $a, b_1, b_2 \in L$.

(ii) A t -norm T on a complete lattice L is called infinitely $\vee$ -distributive if

$$T(a, \bigvee_Q b_\tau) = \bigvee_Q T(a, b_\tau) \quad (3.2)$$

for every subset $\{a, b_\tau \preceq L, \tau \in Q\}$ of L .

Theorem 3.2. [7]

If T is an infinitely $\vee$ -distributive t -norm, then $\bigvee_T \{a_\tau \mid \tau \in Q\} = \bigvee \{a_\tau \mid \tau \in Q\}$ for every $\{a_\tau \mid \tau \in Q\} \subseteq H_T$ and $(H_T, \preceq_T)$ is a complete lattice.

Proof.

For every Q , let us show that $\bigvee_T \{a_\tau \mid \tau \in Q\} = \bigvee \{a_\tau \mid \tau \in Q\}$, where $\{a_\tau \mid \tau \in Q\} \subseteq H_T$. Since, for every $\tau \in Q$, $a_\tau = T(a_\tau, a_\tau) \leq T(\bigvee_{\tau \in Q} a_\tau, \bigvee_{\tau \in Q} a_\tau)$, $\bigvee_{\tau \in Q} a_\tau \leq T(\bigvee_{\tau \in Q} a_\tau, \bigvee_{\tau \in Q} a_\tau) \leq \bigvee_{\tau \in Q} a_\tau$. Then, we have that for every $\tau \in Q$, $\bigvee_{\tau \in Q} a_\tau \in H_T$. Moreover, $\bigvee_{\tau \in Q} a_\tau \in \overline{\{a_\tau \mid \tau \in Q\}}_T$. Indeed, since $a_\tau = T(a_\tau, a_\tau) \leq T(\bigvee_{\tau \in Q} a_\tau, a_\tau) \leq a_\tau$; i.e., $T(\bigvee_{\tau \in Q} a_\tau, a_\tau) = a_\tau$, and so $a_\tau \preceq_T \bigvee_{\tau \in Q} a_\tau$. Thus, $\bigvee_{\tau \in Q} a_\tau \in \overline{\{a_\tau \mid \tau \in Q\}}_T$. Let k be an arbitrary element of the set $\overline{\{a_\tau \mid \tau \in Q\}}_T$. Then, $a_\tau \preceq_T k$. So, there exists the subset $\{b_\tau \mid \tau \in Q\} \subseteq L$ such that $a_\tau = T(k, b_\tau)$. Since T is an infinitely $\vee$ -distributive t -norm, $\bigvee_{\tau \in Q} a_\tau = \bigvee_{\tau \in Q} T(k, b_\tau) = T(k, \bigvee_{\tau \in Q} b_\tau)$. Therefore, $\bigvee_{\tau \in Q} a_\tau \preceq_T k$. This shows that $\bigvee_T \{a_\tau \mid \tau \in Q\} = \bigvee \{a_\tau \mid \tau \in Q\}$. By $0 \in H_T$, $(H_T, \preceq_T)$ is a complete lattice. $\square$

The following corollary follows from the proof of **Theorem 3.1** and **Theorem 3.2**.

Corollary 3.2. [7]

If L is a bounded lattice and T is a $\bigvee$ -distributive t -norm on L , then it is easily obtained that $a \wedge_T b = T(a, b)$ and $a \vee_T b = a \vee b$ for $a, b \in H_T$. Thus, $(H_T, \preceq_T)$ is a bounded lattice.

Corollary 3.3.

If T is an infinitely $\bigvee$ -distributive t -norm on L , then $(H_T, \preceq_T)$ is a Heyting algebra (see [7], p. 273, for the definition of Heyting algebra).

Proof.

For $a \in H_T$ and $\{b_\tau \mid \tau \in Q\} \subseteq H_T$, since T is an infinitely $\bigvee$ -distributive t -norm, $T(a, \bigvee_{\tau \in Q} b_\tau) = \bigvee_{\tau \in Q} T(a, b_\tau)$. By **Theorem 3.2**, since $T(a, \bigvee_{\tau \in Q} b_\tau) = a \wedge_T (\bigvee_{\tau \in Q} b_\tau)$ and $\bigvee_{\tau \in Q} T(a, b_\tau) = \bigvee_{\tau \in Q} (a \wedge_T b_\tau)$, we obtain that $a \wedge_T (\bigvee_{\tau \in Q} b_\tau) = \bigvee_{\tau \in Q} (a \wedge_T b_\tau)$. Thus $(H_T, \preceq_T)$ is a Heyting algebra. $\square$

Corollary 3.4.

Let L be a complete lattice, T be an infinitely $\bigvee$ -distributive and divisible t -norm. Then, $(H_T, \leq)$ is a Heyting algebra.

Proof.

Since T is a divisible t -norm, we have that $\leq = \preceq_T$. For any $a, b \in H_T$, $a \wedge b \preceq_T a$ and $a \wedge b \preceq_T b$, whence $a \wedge b \preceq_T a \wedge_T b$. On the other hand, since $a \wedge_T b = T(a, b) \leq a \wedge b$, it holds that $T(a, b) = a \wedge b$, for $a, b \in H_T$. $\square$

Corollary 3.5.

Let L be a bounded lattice and T be a $\bigvee$ -distributive and divisible t -norm. Then, $(H_T, \leq)$ is a distributive lattice.

Let L be a complete lattice, T be a t -norm on L and $L_1 \subseteq L$. The notation $T \downarrow L_1$ will be used for the restriction of T to L_1 .

Proposition 3.8.

Let $(L, \leq)$ be a bounded lattice and T be a t -norm on L . Then $T \downarrow H_T$ is a t -norm on $(H_T, \preceq_T)$.

Proof.

For every $a, b \in H_T$, we must show that $T(a, b) \in H_T$. Making use associativity of t-norm T , we have that

$$\begin{aligned}
T(T(a, b), T(a, b)) &= T(T(T(a, b), a), b) \\
&= T(T(T(b, a), a), b) \\
&= T(T(b, T(a, a)), b) \\
&= T(T(b, a), b) \\
&= T(T(a, b), b) \\
&= T(a, T(b, b)) \\
&= T(a, b).
\end{aligned}$$

Then, $T(a, b)$ is an element of H_T . From **Proposition 3.5** it is obvious that the monotonicity of T with respect to $\preceq_T$ and associativity. Since $1 \in H_T$, we have that $T(x, 1) = x$ for every $x \in H_T$. Therefore, $T \downarrow H_T$ is a t-norm on H_T . $\square$

Proposition 3.9.

Let T be a t-norm on a bounded lattice L and $K \subseteq L$ be a lattice with respect to the order on L . If $x \wedge_T y = T(x, y)$ for every $x, y \in K$, then $T \downarrow K = \wedge$. Especially, if $K = L$, then $T = \wedge$.

Proof.

Since, for every $x \in K$, $x \wedge_T x = x = T(x, x)$, we obtain that $x \wedge y = T(x \wedge y, x \wedge y) \leq T(x, y) \leq x \wedge y$, for every $x, y \in K$. Therefore, $T(x, y) = x \wedge y$. $\square$

Corollary 3.6. (*Mitsch [20]*)

Let $L = [0, 1]$ and T be any t-norm on L . Then $(H_T, \preceq_T) = (H_T, \leq)$

Conclusion

In this memory, we have studied some algebraic, analytic and logic aspects of triangular norms. As application, we have studied triangular orders (t-order for short), they are partial orders on a bounded lattice obtained from a t-norm on this bounded lattice.

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